11/30

Lecture 20

HW 3 Due Thurs

Website Updated

- Recitation 6 Online (Solving RR)
- Possibly 1 More HW Due After final
- Exam 3 on 12/9, Review on 12/7
- Recitation 7 Soon

Probability

Axioms

$0 \le p(x_i) \le 1$

$\sum_{i=1}^{n} p(x_i) = 1$

p is called a prob dist.

1) The prob of an event is a non-neg real number $P(E) \in \mathbb{R}$, $P(E) \ge 0 \ \forall E \in F$ (event space)

2) $P(\Omega) = 1$ the prob at least one of the elementary events in SS in entire SS occur is 1

3) Any countable seq of disjoint events (mutually exclusive) E_1, E_2 satisfies

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

As a result

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$ Addition Law of Prob

Inclusion-Exclusion Principle is an extension of Addition Law

$$P(A^c) = P(\Omega \setminus A) = 1 - P(A)$$

The prob that an event will not happen is 1 - prob that it will happen

Def The unif dist on a set S w/ n elements assigns prob 1/n to each element of S.

The prob of an event is the sum of the probs of the outcomes in E

$$P(E) = \sum_{s \in E} p(s)$$

Conditional Prob

What if we flip a coin 3 times and all 8 possibilities equally likely?

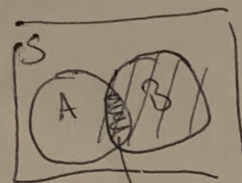
If the first flip is tails, now only four possible outcomes → event F

What is the prob of an odd # of tails? → event E

TTH
TTT -
THT
THH -

HHH
HHT -
HTH -
HTT
THH. -
THT
TTH
TTT -

2/4



P(A|B)

Def Let E, F be events w/ $P(F) > 0$.
The cond prob of $P(E \mid F) = \frac{P(E \wedge F)}{P(F)}$

⇒ 4/8 P(odd tails) = 4/8

Def Events E, F indep iff $P(E, F) = P(E)P(F)$

Events $E_1, \ldots, E_n$ pairwise indep iff $P(E_i \wedge E_j) = P(E_i)P(E_j)$ $\forall i, j$ pairs $1 \le i < j \le n$

Mutually indep iff $P(E_{i1} \wedge E_{i2} \wedge E_{i3} \ldots E_{im})$
$= P(E_{i1})P(E_{i2}) \ldots P(E_{im})$

whenever $i, j = 1, 2, m$ integers w/ $1 < i_1 < i_2 \ldots < i_m \le n$
$m \ge 2$

ernoulli Trials & Binomial Dist

Sps an exp has only two possible outcomes. Ex, flipping a coin, generating a bit at random.

Each performance of such exp is called a Bernoulli Trial.

if p prob success, q prob failure $\Rightarrow p+q=1$

Bernoulli Trials are mutually indep.

Ex A coin biased so prob heads is 2/3. what is prob 4 heads when flipped 7 times?

~~Ex~~: 2^7 possible outcomes $=128$

$$\binom{7}{4}\left(\frac{2}{3}\right)^{\cancel{3}4}\left(\frac{1}{3}\right)^{4}$$

$\Rightarrow$ The prob of ~~success~~ exactly k successes in n bernoulli trials w/ prob success p, failure q is

$$b(k,n,p) = \binom{n}{k} p^{k} (1-p)^{n-k}$$

The Binomial Dist

$$\sum_{k=0}^{n} \cancel{C(n,k)} \binom{n}{k} p^{k}(1-p)^{n-k} = (p+(1-p))^{n} = 1$$

RVs

often interested in a number (numerical value) associated w/ outcome of exp.

Ex, total # heads when coin flipped 20 times.

A RV is a function from SS of exp to set of real numbers.

$\Rightarrow$ A RV is a function, not a variable, not random.

Ex. Coin flipped 3 times. Let $X(t)$ be RV equal to # heads that is the outcome.

$X(HHH) = 3$

$X(HHT) = X(HTH) = X(THH) = 2$

$X(TTH) = X(THT) = X(HTT) = 1$

$X(TTT) = 0$

The dist of the RV X on SS S is set of pairs $(r, p(X=r))\ \forall r \in X(S)$

where $P(X=r)$ is prob that X takes value r.

The set of pairs in the dist is defined by probs $P(X=r)$ for $r \in X(S)$.

Ex. Dist of RV $X(t)$ in above is $(3, 1/8), \ldots, (0, 1/8)$

$P(X=3) = 1/8$

$P(X=2) = 3/8$

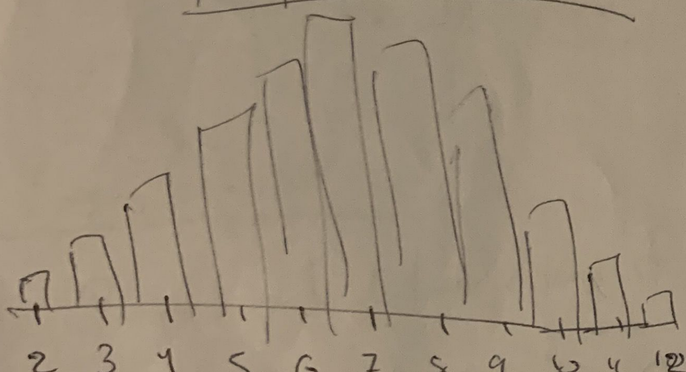
$P(X=1) = 3/8$

$P(X=0) = 1/8$

Ex X be sum of numbers on two dice.

what are values of RV and dist of X?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12



$P(X=2) = 1/36$

$P(X=3) = 2/36$

$P(X=4) = 3/36$

$P(X=5) = 4/36$

$P(X=6) = 5/36$

$P(X=7) = 6/36$

$P(X=8) = 5/36$

$P(X=9) = 4/36$

$P(X=10) = 3/36$

$P(X=11) = 2/36$

$P(X=12) = 1/36$

Bayes Thm.

Can derive a result using def of joint & conditional prob

$$P(A,B) = P(B,A)$$

$$P(A|B) = \frac{P(A,B)}{P(B)} \Rightarrow \begin{array}{l} P(A,B) = P(A|B)P(B) \\ P(B,A) = P(B|A)P(A) \end{array}$$

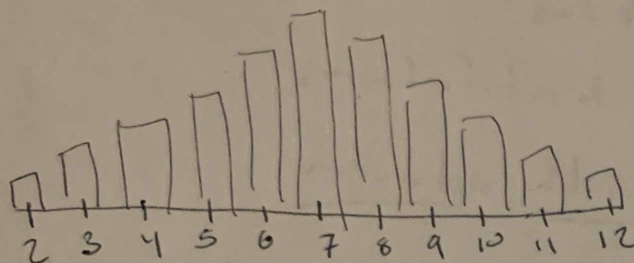
$$\Rightarrow P(A|B)P(B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_A P(B|A)P(A)} = \frac{P(B|A)P(A)}{\sum_A P(B,A)}$$

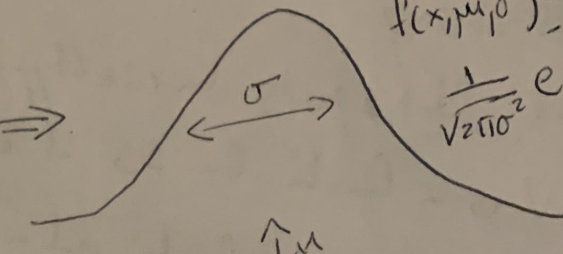
Remark

Histogram / Dist. of Sum of two Indep Die

looks a lot like bell curve



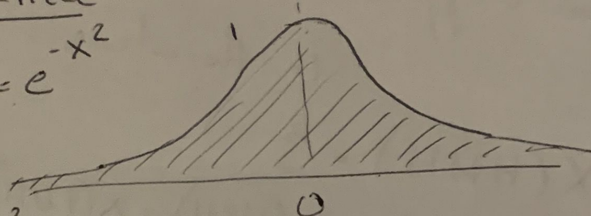
$\Rightarrow$



$$f(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

The Central Limit Thm in statistics states that when indep. RV's are summed, the normalized sum tends toward a Normal dist

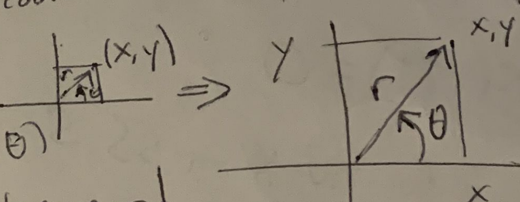
Lets look at the function $f(x) = e^{-x^2}$



How is it normalized?

$$A = \int_{-\infty}^{\infty} e^{-x^2} dx \Rightarrow A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx\,dy$$

$$\Rightarrow = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy.$$ Any pt in x, y coord can be represented w/ radius r and angle θ



$x = r\cos\theta$
$y = r\sin\theta$

to convert from (x,y) to (r,θ) we need to look at the determinant of the jacobian

$$\begin{vmatrix} \frac{dx}{dr} & \frac{dy}{dr} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - -r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

So we pick up a factor of r in the integrand when changing coordinates

$$\Rightarrow A^2 = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr\,d\theta.$$ Try u substitution $u = -r^2$ $du = -2r\,dr$

$\Rightarrow r\,dr = -\frac{1}{2} du$, $u(0) = 0$
$u(+\infty) = -\infty$

$$= \int_0^{2\pi}\int_0^{-\infty} -\frac{1}{2} e^{u} du\,d\theta = \int -\frac{1}{2} e^{u}\Big|_0^{-\infty} d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

$\Rightarrow A = \sqrt{\pi}$ which allows us to define the Normal Dist

$$f(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Def Marginal Prob

prob an individual event from exp occurs regardless of outcomes from other exps

- computed by adding probs across row/col of event
- written in margins of table

		Vote for X		
		Y	N	Total
Gender	M	.41	.07	.48
	F	.47	.05	.52
		.88	.12	1.00

Bayes Thm Ex

Patient prob of liver disease if alcoholic

$P(\text{Liver disease}) = P(A) = .10$

$P(\text{Alcoholic}) = P(B) = .05$

$P(\text{Alcoholic} | \text{Liver}) = P(B|A) = .07$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)} = \frac{.07 \times .1}{.05} = \frac{.007}{.05} = .14$$

If Alcoholic, chance of liver disease = 14%

Moments

Interested in average value of RV called expected value

$$E(X) = \sum_{s \in S} p(s)X(s)$$

deviation $\quad X(s) - E(X)$

$$E(X) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \dots + \frac{1}{6}\cdot 6 = \sum_{i=1}^{6} \frac{1}{6}\cdot i = \frac{21}{6} = \frac{7}{2}$$

if X an RV and $P(X=r)$ prob $X=r$

$$E(X) = \sum_{r \in X(S)} P(X=r)\,r$$

~~Ex/ X be r.v. # successes in n bernoulli trials~~

$$\cancel{P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

$$\cancel{E(X) = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}}$$

$E(X_1 + X_2)$

$= \sum p(s)(X_1(s) + X_2(s))$

$= \sum p(s) X_1(s) + \sum p(s) X_2(s)$

$= E(X_1) + E(X_2)$

$E(aX+b) = \sum p(s)(aX(s)+b)$

$= a \sum p(s) X(s) + \sum p(s) b$

$= a\,E(X) + b$

Linearity of expectation

if X_i, $i = 1, \dots, n$ are RVs on S, and $a, b \in \mathbb{R}$

$$E(X_1 + X_2 + \dots X_n) = E(X_1) + E(X_2) + E(X_n)$$

$$E(aX+b) = a\,E(X) + b$$

Geometric Dist

Sps prob tails is p.

Coin flipped repeatedly until tails. What is expected # flips?

SS: $\cancel{T/H}$ $\cancel{HT}$ T, HT, HHT, HHHT, ...

$\cancel{TT\ TH}$

$\cancel{TTT\ TH}$

infinite SS.

$P(T) = p$

$P(H\cancel{T}) = (1-p)p$ $\Rightarrow$ prob coin flipped

$P(X=n) =$ n Leg

$P(HHT) = (1-p)^2 p$ $\Rightarrow (1-p)^{n-1} \cdot p$

$$\sum_{n=1}^{\infty} (1-p)^{n-1} p = \cancel{\frac{X}{1-(1-p)}} = \quad p\cancel{(1-)}$$

$$= p \sum_{n=1}^{\infty} (1-p)^{n-1} = p \sum_{k=0}^{\infty} (1-p)^k = p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

$$E(X) = \sum_{j=1}^{\infty} j \cdot P(X=j) = \sum_{j=1}^{\infty} j (1-p)^{j-1} p = p \sum_{j=1}^{\infty} j (1-p)^{j-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

$$= \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$